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The nonsymmetric-nonabelian Kaluza-Klein theory

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Abstract. This paper is devoted to an $(n+4)$-dimensional unification of Moffat's theory of gravitation and Yang-Mills field theory with nonabelian gauge group G. We found 'interference effects' between gravitational and Yang-Mills (gauge) fields which appear to be due to the skewsymmetric part of the metric of Moffat's theory and the skewsymmetric part of the metric on the group G. Our unification, called the nonsymmetric-nonabelian Kaluza-Klein theory, becomes classical Kaluza-Klein theory if the skewsymmetric parts of both metrics are zero.

1. Introduction

The aim of this paper is to find the Kaluza-Klein analogue for Moffat's theory of gravitation (Moffat 1979, 1981, 1982a). In other words it will be an $(n+4)$-dimensional unification of Moffat's theory of gravitation and a nonabelian gauge theory (Yang-Mills field theory, $n=\dim G$, where G is a gauge group). Our unification called the nonsymmetric-nonabelian Kaluza-Klein theory is analogous to the relation between Moffat's theory and general relativity. The diagram (figure 1) places our unification among the above mentioned theories.

Roughly speaking, in the general theory of relativity, mass curves space-time. In Moffat's theory of gravitation, mass and fermion charge (fermion number) curve and twist space-time. In classical (nonabelian) Kaluza-Klein theory, mass curves space-time, and colour (isotopic) charges curve the additional n dimensions. In the nonsymmetric-nonabelian Kaluza-Klein theory, mass and fermion number curve and twist space-time, and colour (isotopic) charges curve and twist the additional n dimensions.

Moffat's theory of gravitation is based on three fundamental geometrical quantities: two connections $\bar{\Gamma}^{\alpha}_{\beta\gamma}$ and $\bar{W}^{\alpha}_{\beta\gamma}$ and the nonsymmetric metric $g_{\alpha\beta}$. This nonsymmetric metric is equivalent to the existence of two geometrical objects defined on space-time: the symmetric metric tensor

$$\bar{g} = g_{(\alpha\beta)}\bar{\theta}^{\alpha}\otimes\bar{\theta}^{\beta}$$

and the 2-form

$$\underset{\sim}{g} = g_{[\mu\nu]}\bar{\theta}^{\mu}\wedge\bar{\theta}^{\nu}.$$

In the general theory of relativity we have only one connection with vanishing torsion and a symmetric metric on space-time. Thus we have only $\bar{\Gamma}$ and $\bar{g}$. Of course in Moffat's theory connections $\bar{\Gamma}$ and $\bar{W}$ are interrelated and have nonvanishing torsion.

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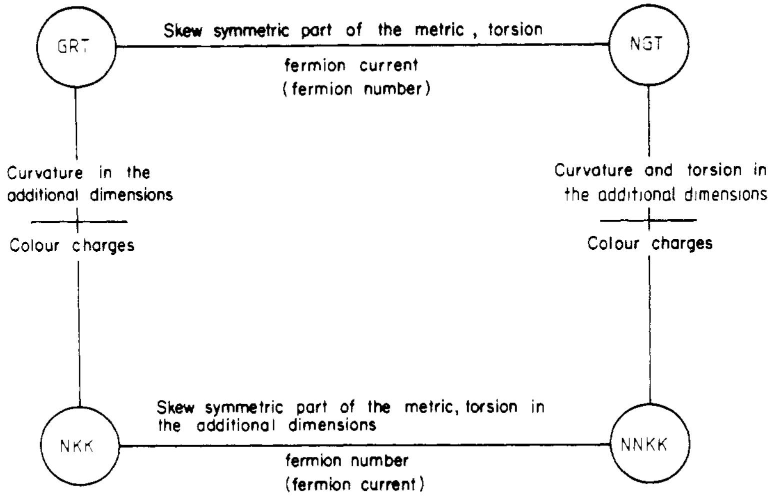

Figure 1. The position of the nonsymmetric–nonabelian Kaluza–Klein theory in general relativity, the nonsymmetric theory of gravitation and the classical nonabelian Kaluza–Klein theory. Abbreviations: GRT, general theory of relativity; NGT, nonsymmetric theory of gravitation (Moffat's theory, real version); NKK, nonabelian Kaluza–Klein theory; NNKK, nonsymmetric–nonabelian Kaluza–Klein theory.

The classical Kaluza–Klein approach and its generalisation to nonabelian gauge groups (Cho 1975, Kaluza 1921, Lichnerowicz 1955b, Kerner 1968, Rayski 1965, Kalinowski 1981, 1983) was based on the following ideas.

On the space–time we have Riemannian geometry based on the metric tensor $\underset{\sim}{g}$ and we have general relativity with the local coordinate invariance principle. Simultaneously, we have a principal fibre bundle over space–time with the structural group G (in the electromagnetic case U(1)). The connection on this bundle describes the Yang–Mills field (gauge field). We have also the local gauge invariance principle for the Yang–Mills fields (or for the electromagnetic field in the case if $G = \mathrm{U}(1)$).

The local coordinate invariance principle and the local gauge invariance principle seem to be two major concepts of physics. The first is basic for general relativity and other alternative theories of gravitation such as Einstein–Cartan theory, Brans–Dicke theory etc. The local coordinate invariance principle is also basic in Moffat's theory of gravitation. The second one, the local gauge invariance principle, is fundamental for electrodynamics. The first was introduced by A Einstein and the second by H Weyl. Now we know the principle of local gauge invariance is fundamental also for weak and strong interactions (Weinberg–Salam model and quantum chromodynamics), but the gauge groups are nonabelian. In the grand unified theories based on some nonabelian groups, the local gauge invariance principle also plays the fundamental role.

The Kaluza–Klein theory unifies these two concepts and reduces them to the first, the local coordinate invariance principle, but in a more than four-dimensional world. In the electromagnetic case we deal with a five-dimensional manifold. In general we deal with an $(n+4)$-dimensional manifold for an arbitrary gauge group, where $n = \dim G$.

The basic idea is very simple. On the gauge group we have a bi-invariant tensor (for example the Cartan–Killing tensor). This tensor plays the role of a metric in the

Lie algebra of the gauge group G (normally it is supposed that G is semisimple). In the abelian five-dimensional case we have as this tensor the number (-1).

On the fibre bundle we have the natural distribution of horizontal spaces induced by the connection. The metric tensor $\bar{g}$ acts on space–time.

We can divide every tangent vector to the fibre bundle in only one way (the connection is established) into two parts—horizontal and vertical. The horizontal part we can project onto space–time and the vertical one, due to the connection, onto the Lie algebra of the gauge group. Thus we have natural (symmetric) metrisation of the fibre bundle. We can 'measure' independently the length of both parts by two (symmetric) metric tensors and then add these two results. The procedure is similar in spirit to Pythagoras's theorem. This construction was first introduced by Trautman (1970). Having the principle bundle metrised in a natural way (the metric tensor is bi-invariant with respect to the gauge group action on the bundle) we introduce linear connections on the bundle (i.e. connections on the bundle of linear frames over the previous principal bundle) which are compatible in some sense with the metric. The simplest solution is to suppose that this connection is the Levi–Civita connection as in the five-dimensional Kaluza–Klein theory. If we calculate the Ricci curvature scalar for this connection we get a sum of the Ricci curvature scalar on space–time and the electromagnetic Lagrangian. In the non-abelian case, $(n+4)$-dimensional, the result will be more complex; we get a sum of the Ricci curvature scalar on space–time, the Yang–Mills Lagrangian plus the cosmological constant which is 10^{127} times bigger than the upper limit from observational data. This makes us change geometry on the metrised fibre bundle, and abandon the Levi–Civita (Riemannian) connection. We must employ the connection with torsion. This was done in a natural geometrical way by Kalinowski (1983); the cosmological constant vanishes (it is almost zero from observational data). In the light of the new observational data concerning the quadrupole moment of mass for the sun (see Hill *et al* 1982) it seems that the general theory of relativity is unable to explain the perihelion movement of Mercury and Icarus.

Moffat's theory can explain observational data (see Moffat 1982b, 1983). Moffat's theory due to using fermion current (fermion number $F = B - L$, B—baryon charge, L—lepton charge) as a second gravitational charge (the first is the mass) seems to be closer to elementary particle theory than general relativity. The fermion charge is conserved in Moffat's theory. In the grand unified theory (see Langacker 1981) based on SO(10), fermion number (fermion charge) is one of the generators of the Lie algebra of SO(10).

Thus it would appear to be important to find the Kaluza–Klein analogue for Moffat's theory in the general nonabelian case in order to carry out further investigations.

This theory, the nonsymmetric–nonabelian Kaluza–Klein theory, unifies the coordinate invariance principle from Moffat's theory and the local gauge invariance principle for Yang–Mills (gauge) fields.

Following ideas concerning the geometry of the Kaluza–Klein theory described above, it is necessary to find the natural nonsymmetric metrisation of the fibre bundle over space–time. The existence of such a nonsymmetric metric on the fibre bundle is equivalent to the existence of two bi-invariant geometrical objects $\bar{\gamma}$ and $\underset{\sim}{\gamma}$. The first $\bar{\gamma}$ is a symmetric bi-invariant tensor and the second $\underset{\sim}{\gamma}$ is a bi-invariant 2-form on the fibre bundle. The first is constructed and used in the classical Kaluza–Klein theory (natural symmetric metrisation). It is necessary to construct the second one.

Following the basic idea of the previous construction it is necessary to choose a bi-invariant skewsymmetric form on the gauge group G. We have a natural skewsymmetric form defined on the Lie algebra of the group. It is the commutator. This form has values in the Lie algebra of the group. But the inner product of this form and the vector $C = h^{ab}\,\mathrm{Tr}[(X_a)^2]X_b$ (trace with respect to the space of the generator representation) is a number. If the representation is real we have what we were looking for. The form is bi-invariant with respect to the group action. Unfortunately this form is often zero. For example it is zero for U(1).

Now following the idea of the symmetric metrisation of the fibre bundle we can build γ from $\underline{g}$ and the above form. If the form is zero $\gamma = \pi^*(\underline{g})$, where π^* is a pull-back of π (the natural projection on the fibre bundle). This is in the electromagnetic case and it was done in Kalinowski (1982).

The nonsymmetric–nonabelian Kaluza–Klein theory seems to be a real unified theory of Yang–Mills and gravitational fields. It not only reduces two major principles of invariance to the local coordinate invariance principle, but it provides new effects, which are absent in the classical Kaluza–Klein theory. These effects are also absent in Moffat's theory of gravitation and in Yang–Mills theory and are thus 'interference effects' between gravitational and Yang–Mills fields. They are as follows.

(i) The new term in the Yang–Mills Lagrangian

$$(1/4\pi)h_{ab}(g^{[\mu\nu]}H^a_{\mu\nu})(g^{[\alpha\beta]}H^b_{\alpha\beta}).$$

(ii) The change in the classical part of the Yang–Mills Lagrangian in replacing h_{ab} by

$$l_{ab} = h_{ab} + \mu K_{ab}.$$

(iii) The existence of a Yang–Mills field polarisation of the vacuum, $M^a{}_{\alpha\beta}$.

(iv) The additional term in the Kerner–Wong equation (equation of motion for the test particle in the gravitational and Yang–Mills fields)

$$\tfrac{1}{2}(q^b/m_0)(l_{bd}g^{\alpha\beta} - l_{db}g^{\beta\alpha})L^d_{\beta\gamma}U^\gamma$$

where m_0 is a rest mass of a test particle and q^b is its colour (isotopic) charge.

(v) The existence of the cosmological constant $\rho(\mu)$ with asymptotic behaviour for large μ

$$\rho(\mu) \sim \mathrm{constant}/\mu^2.$$

(vi) The new energy–momentum tensor $\overset{\text{gauge}}{T}_{\alpha\beta}$ with zero trace.

(vii) Sources for Yang–Mills fields, the current $\tilde{j}^{\alpha a}$.

All of these effects vanish if the metric of $\underline{P}$ (fibre bundle) becomes symmetric. In this case we get the classical symmetric nonabelian Kaluza–Klein theory.

The paper is organised as follows. In § 2 we introduce the notations and definitions of all geometrical quantities which we use throughout the paper. In § 3 we define the natural nonsymmetric metrisation of the principal fibre bundle. In § 4 we formulate the nonsymmetric–nonabelian Kaluza–Klein theory. In § 5 we write down the geodesic equation on $\underline{P}$ (nonsymmetrically metrised fibre bundle) and we find a new correction for the Kerner–Wong equation. We calculate connections $\omega^A{}_B$ and $W^A{}_B$ which are analogous to connections $\bar{\omega}^\alpha{}_\beta$ and $\bar{W}^\alpha{}_\beta$. In § 6 we calculate the 2-form of torsion and the 2-form of curvature for the connection $\omega^A{}_B$. After this we write the curvature tensor for $\omega^A{}_B$ and the Moffat–Ricci curvature scalar for $\omega^A{}_B$. Using the results obtained we calculate the Moffat–Ricci curvature scalar for the connection $W^A{}_B$. In § 7 we deal with the connection $\tilde{\omega}^a{}_b$ defined on the fibre and with cosmological constant which is the Moffat–Ricci curvature scalar for $\tilde{\omega}^a{}_b$. We find the asymptotic properties of the cosmological constant for very large μ. In § 8 we define the Palatini variational principle for the Moffat–Ricci curvature scalar $R(W)$ and obtain field equations for gravitational and Yang–Mills fields. We discuss and interpret our results and point out all the differences between the classical and the nonsymmetric–nonabelian Kaluza–Klein theory. We write down all 'interference effects' between gravitational and Yang–Mills fields which appear in our theory. In § 9 we deal with some special cases of our theory.

2. Elements of geometry

In this section we introduce the notations and define geometric quantities used in the paper. We use a smooth principal fibre bundle $\underline{P}$, which includes in its definition the following list of differentiable manifolds and smooth maps:

a total (bundle) space $\underline{P}$,

a base E; in our case it is a space–time,

a projection $\pi: \underline{P} \to E$,

a map $\Phi: \underline{P} \times G \to \underline{P}$ defining the action of G on $\underline{P}$; if $a, b \in G$ and $\varepsilon \in G$ is the unit element then $\Phi(a) \circ \Phi(b) = \Phi(ba)$ and $\Phi(\varepsilon) = id$ and $\Phi(a)p = \Phi(p, a)$, moreover $\pi \circ \Phi(a) = \pi$. ω is a 1-form of a connection on $\underline{P}$ with values in the Lie algebra of the group G.

Let $\Phi'(a)$ be the tangent map to $\Phi(a)$ whereas $\Phi^*(a)$ is contragredient to $\Phi(a)$ at the point a. The form ω is a form of ad-type i.e.:

$$\Phi^*(a)\omega = ad'_{a^{-1}}\omega \tag{2.1}$$

where $ad'_{a^{-1}}$ is the tangent map to the internal automorphism of the group G and $ad_a(b) = aba^{-1}$. Due to the form ω we can introduce the distribution field of linear elements H_r, $r \in \underline{P}$, where $H_r \subset T_r(\underline{P})$ is a subspace of the space tangent to $\underline{P}$ at a point r and

$$v \in H_r \Leftrightarrow \omega(v) = 0. \tag{2.2}$$

We have

$$T_r(\underline{P}) = V_r \oplus H_r \tag{2.3}$$

where H_r is called a subspace of horizontal vectors and V_r of vertical vectors. For vertical vectors $v \in V_r$ we have $\pi'(v) = 0$. This means that v is tangent to fibres. Let us define

$$v = \mathrm{hor}(v) + \mathrm{ver}(v), \qquad \mathrm{hor}(v) \in H_r, \qquad \mathrm{ver}(v) \in V_r. \tag{2.4}$$

It is well known that the distribution H_r is equivalent to a choice of the connection ω. We use the operation 'hor' for forms, i.e.

$$(\mathrm{hor}\,\beta)(X, Y) = \beta(\mathrm{hor}\,X, \mathrm{hor}\,Y) \tag{2.5}$$

where $X, Y \in T_r(\underline{P})$. The 2-form of curvature of the connection ω is:

$$\Omega = \mathrm{hor}\; \mathrm{d}\omega. \tag{2.6}$$

It is also a form of ad-type like ω. Ω obeys the structural Cartan equation

$$\Omega = \mathrm{d}\omega + \tfrac{1}{2}[\omega, \omega] \tag{2.7}$$

where $[\omega, \omega](X, Y) = [\omega(X), \omega(Y)]$. Bianchi's identity for ω is

$$\operatorname{hor} \mathrm{d}\Omega = 0. \tag{2.8}$$

For the principal fibre bundle we use the following convenient scheme (figure 2). The map $e: U \to \underline{P}$, $U \subset E$, so that $e \circ \pi = id$ is called a local section. From the physical point of view it means choosing the gauge. Thus

$$e^*\omega = e^*(\omega^a X_a) = A^a{}_\mu \bar{\theta}^\mu X_a \tag{2.9}$$

$$e^*\Omega = e^*(\Omega^a X_a) = \tfrac{1}{2} F^a_{\mu\nu} \bar{\theta}^\mu \wedge \bar{\theta}^\nu X_a.$$

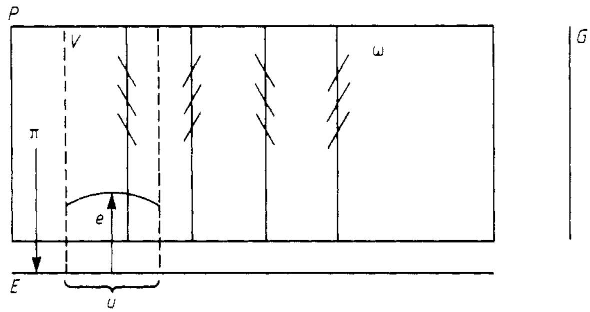

Figure 2. Principal fibre bundle $\underline{P}$.

Further we introduce a notation

$$\Omega^a = \tfrac{1}{2} H^a{}_{\mu\nu} \theta^\mu \wedge \theta^\nu \tag{2.10}$$

where $\theta^\mu = \pi^*(\bar{\theta}^\mu)$, $\bar{\theta}^\mu$ is a basis on E, π^* is contragredient to π and

$F^a_{\mu\nu} = \partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu + C^a_{bc} A^b_\mu A^c_\nu$, $\qquad X_a$, $\qquad a = 1, 2 \ldots \dim G = n$

are generators of the Lie algebra of the group G and

$$[X_a, X_b] = C^c{}_{ab} X_c.$$

In this paper we also use a classical linear connection on manifolds $\underline{P}$ and E using the formalism of differential forms. So the basic quantity is a 1-form of connection $\omega^A{}_B$ (coefficients of the connection). The 2-form of curvature is the following

$$\Omega^A{}_B = \mathrm{d}\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B \tag{2.11}$$

and the 2-form of torsion

$$\theta^A = \mathrm{D}\theta^A \tag{2.12}$$

where θ^A are basic forms, D means the exterior covariant derivative with respect to $\omega^A{}_B$. The following relations define the interrelation between our symbols and the generally used ones

$$\omega^A{}_B = \Gamma^A{}_{BC} \theta^C, \qquad \Theta^A = \tfrac{1}{2} Q^A{}_{BC} \theta^B \wedge \theta^C, \qquad \Omega^A{}_B = \tfrac{1}{2} R^A{}_{BCD} \theta^C \wedge \theta^D, \tag{2.13}$$

where $\Gamma^A{}_{BC}$ are coefficients of the connection (they do not have to be symmetric in indices B and C), $R^A{}_{BCD}$ is a tensor of curvature and $Q^A{}_{BC}$ is a tensor of torsion. Covariant exterior differentiation with respect to $\omega^A{}_B$ is given by the formulae

$$\begin{aligned} \mathrm{D}\Xi^A &= \mathrm{d}\Xi^A + \omega^A{}_C \wedge \Xi^C \\ \mathrm{D}\Sigma^A{}_B &= \mathrm{d}\Sigma^A{}_B + \omega^A{}_C \wedge \Sigma^C{}_B - \omega^C{}_B \wedge \Sigma^A{}_C. \end{aligned} \tag{2.14}$$

The form of curvature $\Omega^A{}_B$ and torsion Θ^A obey Bianchi's identities

$$\mathrm{D}\Omega^A{}_B = 0, \qquad \mathrm{D}\Theta^A = \Omega^A{}_B \wedge \theta^B. \tag{2.15}$$

In the paper we also use Einstein's + and − differentiations for the nonsymmetric metric tensor g_{AB}

$$\mathrm{D}g_{A+B-} = \mathrm{D}g_{AB} - g_{AD}Q^D{}_{BC}\theta^C \tag{2.16}$$

where D is the covariant exterior derivative with respect to $\omega^A{}_B$ and $Q^D{}_{BC}$ is the tensor of torsion for $\omega^A{}_B$. In a holonomic system of coordinates we easily get:

$$\mathrm{D}g_{A+B-} = g_{A+B-;C}\theta^C = (g_{AB,C} - g_{DB}\Gamma^D_{AC} - g_{AD}\Gamma^D_{CB})\theta^C. \tag{2.17}$$

All quantities introduced in this section and their precise definitions can be found in Trautman (1970), Kobayashi and Nomizu (1963), Lichnerowicz (1955a) and Einstein (1953).

3. The natural nonsymmetric metrisation of a bundle $\underline{P}$

Let us introduce the principal fibre bundle P over the space–time E with the structural group G and with the projection π. Let us suppose that (E, g) is a manifold with nonsymmetric metric tensor

$$g_{\mu\nu} = g_{(\mu\nu)} + g_{[\mu\nu]}. \tag{3.1}$$

Let us introduce a horizontal lift basis on $\underline{P}$.

$$\theta^A = (\pi^*(\bar{\theta}^\alpha), \theta^a = \lambda\omega^a), \qquad \lambda = \text{constant}. \tag{3.2}$$

It is convenient to introduce the following notations. Capital indices A, B, C, run $1, 2, 3, \ldots, n+4$, $n = \dim G$. Lower case greek indices $\alpha, \beta, \gamma, \delta = 1, 2, 3, 4$ and lower case latin $a, b, c, d = 5, 6 \ldots n+4$. An overbar above θ^α and other quantities indicates that they are defined on E.

It is easy to see that the existence of the nonsymmetric metric on E is equivalent to introducing two independent geometrical quantities on E

$$\bar{g} = g_{\alpha\beta}\bar{\theta}^\alpha \otimes \bar{\theta}^\beta = g_{(\alpha\beta)}\bar{\theta}^\alpha \otimes \bar{\theta}^\beta \tag{3.3}$$

$$\underline{g} = g_{\alpha\beta}\bar{\theta}^\alpha \wedge \bar{\theta}^\beta = g_{[\alpha\beta]}\bar{\theta}^\alpha \wedge \bar{\theta}^\beta \tag{3.4}$$

i.e. the symmetric metric tensor $\bar{g}$ on E and the 2-form $\underline{g}$. On the group G we can introduce a bi-invariant symmetric tensor called the Killing–Cartan tensor

$$h(A, B) = \mathrm{Tr}(Ad_A \circ Ad_B) \tag{3.5}$$

where $Ad_A(C) = [A, C]$. It is easy to see that

$$h(A, B) = h_{ab}A^a \cdot B^b \tag{3.6}$$

where $h_{ab}=C^c{}_{ad}C^d{}_{bc}$, $h_{ab}=h_{ba}$, $A=A^aX_a$, $B=B^aX_a$. This tensor is distinguished by the group structure, but there are of course other bi-invariant tensors on G. Normally it is supposed that G is semisimple. It means that $\det(h_{ab})\neq 0$. What is a natural 2-form on G, or a natural skewsymmetric bi-invariant tensor? It is easy to see that

$$K(A,B)=h([A,B],C), \qquad C=h^{ab}\,\mathrm{Tr}[(X_a)^2]X_b \tag{3.7}$$

has these properties and

$$K(A,B)=K_{bc}A^b\cdot B^c \tag{3.8}$$

where

$$K_{bc}=C^a{}_{bc}\cdot\mathrm{Tr}[(X_a)^2], \qquad K_{bc}=-K_{cb}.$$

Trace (Tr) is here understood in the sense of the representation space of generators X_a. If the representation is a real, then K is a real too. The tensor K is zero in the following two important cases: (a) G is abelian; (b) $\bigwedge_a \mathrm{Tr}[(X_a)^2]=0$. Thus K is zero for U(1). Let us turn to the nonsymmetric natural metrisation of $\underset{\sim}{P}$. Let us suppose that:

$$\bar{\gamma}(X,Y)=\bar{g}(\pi'X,\pi'Y)+\lambda^2h(\omega(X),\omega(Y)) \tag{3.9}$$

$$\underset{\sim}{\gamma}(X,Y)=\underset{\sim}{g}(\pi'X,\pi'Y)+\mu\lambda^2K(\omega(X),\omega(Y)) \tag{3.10}$$

μ = constant and is dimensionless, $X,Y\in\tan(\underset{\sim}{P})$. The first equation (3.9) was introduced by Trautman (1970) for the symmetric natural metrisation of $\underset{\sim}{P}$ and was used to construct the Kaluza–Klein theory for U(1) and nonabelian generalisations of this theory (Kerner 1968, Cho 1975 and Kalinowski 1983). It is easy to see that

$$\bar{\gamma}=\pi^*\bar{g}+h_{ab}\theta^a\otimes\theta^b \tag{3.11}$$

$$\underset{\sim}{\gamma}=\pi^*\underset{\sim}{g}+\mu K_{ab}\theta^a\wedge\theta^b \tag{3.12}$$

or

$$\gamma_{(AB)}=\begin{pmatrix} g_{(\alpha\beta)} & 0\\ 0 & h_{ab}\end{pmatrix} \tag{3.13}$$

$$\gamma_{[AB]}=\begin{pmatrix} g_{[\alpha\beta]} & 0\\ 0 & \mu K_{ab}\end{pmatrix}. \tag{3.14}$$

For

$$\gamma_{AB}=\gamma_{(AB)}+\gamma_{[AB]}$$

one easily gets

$$\gamma_{AB}=\begin{pmatrix} g_{\alpha\beta} & 0\\ 0 & l_{ab}\end{pmatrix} \tag{3.15}$$

where $l_{ab}=h_{ab}+\mu K_{ab}$. Tensor γ_{AB} has this simple form in the natural frame on $\underset{\sim}{P}$, θ^A. This frame is unholonomical, because:

$$\mathrm{d}\theta^a=(\lambda/2)[H^a{}_{\mu\nu}\theta^\mu\wedge\theta^\nu-(1/\lambda^2)C^a{}_{bc}\theta^b\wedge\theta^c]\neq 0. \tag{3.16}$$

We also introduce a dual frame

$$\bar{\gamma}(\xi_A)=\gamma_{(AB)}\theta^B. \tag{3.17}$$

We have $\xi_A = (\xi_\alpha, \xi_a)$ and

$$\mathscr{L}_{\xi_a} \tilde{\gamma} = 0 \tag{3.18}$$

$$\mathscr{L}_{\xi_a} \underset{\sim}{\gamma} = 0. \tag{3.19}$$

Thus γ is bi-invariant with respect to the group action on $\underset{\sim}{P}$ (ξ_a are of course fundamental fields on $\underset{\sim}{P}$). In the case with $\mathrm{Tr}[(X_a)]^2 = 0$ for every a we have

$$\gamma_{AB} = \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & h_{ab} \end{pmatrix}. \tag{3.20}$$

For the electromagnetic case ($G = \mathrm{U}(1)$) one easily finds

$$\gamma_{AB} = \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.21}$$

Now let us take a section $e: E \to \underset{\sim}{P}$ and fit to it a frame Φ^a, $a = 5, 6 \ldots n+4$, selecting X^μ = constant on a fibre in such a way that e is given by the condition

$$e^*\Phi^a = 0.$$

Thus we have

$$\omega = (1/\lambda)\Phi^a X_a + \pi^*(A^a_\mu \bar{\theta}^\mu) X_a$$

where

$$e^*\omega = A = A^a{}_\mu \bar{\theta}^\mu X_a.$$

In this frame the tensor γ takes the form

$$\gamma_{AB} = \begin{pmatrix} g_{\alpha\beta} + \lambda^2 l_{ab} A^a_\alpha A^b_\beta & \lambda l_{cb} A^c_\alpha \\ \lambda l_{ac} A^c_\beta & l_{ab} \end{pmatrix} \tag{3.22}$$

where

$$l_{ab} = h_{ab} + \mu K_{ab}.$$

This frame is also unholonomic. One easily finds

$$\mathrm{d}\Phi^a = -(1/2\lambda) C^a{}_{bc} \Phi^b \wedge \Phi^c. \tag{3.23}$$

The nonsymmetric theory of gravitation (see Moffat 1979, 1981, 1982a) uses the nonsymmetric metric $g_{\mu\nu}$ such that

$$g_{\mu\nu} g^{\beta\nu} = g_{\nu\mu} g^{\nu\beta} = \delta^\beta_\mu \tag{3.24}$$

where the order of indices is important. If G is semi-simple and $\mathrm{Tr}[(X_a)]^2 = 0$ for every a

$$l_{ab} = h_{ab}, \qquad \det(h_{ab}) \neq 0$$

and

$$h_{ab} h^{bc} = \delta^c_a. \tag{3.25}$$

Thus one easily finds in this case

$$\gamma_{AC} \gamma^{BC} = \gamma_{CA} \gamma^{CB} = \delta^B_A \tag{3.26}$$

where the order of indices is important. We will have the same for the electromagnetic case ($G = \mathrm{U}(1)$). In general if $\det(l_{ab}) \neq 0$ then

$$l_{ab} l^{ac} = l_{ba} l^{ca} = \delta^c_b \tag{3.27}$$

where the order of indices is important. From (3.15) we have (3.26) for the general nonsymmetric metric γ.

4. Formulation of the nonsymmetric Kaluza-Klein theory

Let P be the principal fibre bundle with the structural group G, over space-time E with a projection π and let us define on this bundle a connection ω. Let us suppose that G is semisimple and that its Lie algebra $\mathfrak{g}$ has a real representation such as $\mathrm{Tr}[(X_a)^2]$ which is not equal to zero for every a. Tr is understood here in the sense of the representation space of the Lie algebra $\mathfrak{g}$. On space-time E we define a nonsymmetric metric tensor such that:

$$\begin{aligned} g_{\alpha\beta} &= g_{(\alpha\beta)} + g_{[\alpha\beta]} \\ g_{\alpha\beta} g^{\gamma\beta} &= g_{\beta\alpha} g^{\beta\gamma} = \delta^\gamma_\alpha \end{aligned} \tag{4.1}$$

where the order of indices is important. We define also on E coefficients of two connections

$$\bar{\omega}^\alpha{}_\beta \text{ and } \bar{W}^\alpha{}_\beta, \qquad \bar{\omega}^\alpha{}_\beta = \bar{\Gamma}^\alpha{}_{\beta\gamma} \bar{\theta}^\gamma \tag{4.2}$$

and

$$\bar{W}^\alpha{}_\beta = \bar{W}^\alpha{}_{\beta\gamma} \bar{\theta}^\gamma, \qquad \bar{W}^\alpha{}_\beta = \bar{\omega}^\alpha{}_\beta - \tfrac{2}{3}\delta^\alpha_\beta \bar{W} \tag{4.3}$$

where

$$\bar{W} = \bar{W}_\gamma \bar{\theta}^\gamma = \tfrac{1}{2}(\bar{W}^\sigma{}_{\gamma\sigma} - \bar{W}^\sigma{}_{\sigma\gamma})\bar{\theta}^\gamma.$$

For the connection form $\bar{\omega}^\alpha{}_\beta$ we suppose the following conditions

$$\bar{\mathrm{D}} g_{\alpha+\beta-} = \bar{\mathrm{D}} g_{\alpha\beta} - g_{\alpha\delta} \bar{Q}^\delta_{\beta\gamma}(\bar{\Gamma})\bar{\theta}^\gamma = 0, \qquad \bar{Q}^\alpha_{\beta\alpha}(\bar{\Gamma}) = 0 \tag{4.4}$$

where $\bar{\mathrm{D}}$ is the exterior covariant derivative with respect to $\bar{\omega}^\alpha{}_\beta$ and $\bar{Q}^\alpha_{\beta\gamma}(\bar{\Gamma})$ is the torsion of $\bar{\omega}^\alpha{}_\beta$. Thus we have on space-time E, all quantities from Moffat's theory of gravitation (see Moffat 1979, 1981, 1982a). Now let us turn to the natural nonsymmetric metrisation of the bundle $\underline{P}$. According to § 3 we have

$$\begin{aligned} \bar{\gamma} &= \pi^* \bar{g} + h_{ab}\theta^a \otimes \theta^b = \pi^*(g_{(\alpha\beta)}\bar{\theta}^\alpha \otimes \bar{\theta}^\beta) + h_{ab}\theta^a \otimes \theta^b \\ \underline{\gamma} &= \pi^* \underline{g} + \mu K_{ab}\theta^a \wedge \theta^b = \pi^*(g_{[\alpha\beta]}\bar{\theta}^\alpha \wedge \bar{\theta}^\beta) + \mu K_{ab}\theta^a \wedge \theta^b \end{aligned} \tag{4.5}$$

where $\theta^a = \lambda\omega^a$. From the classical Kaluza-Klein theory (with symmetric metric) we know that $\lambda = 2\sqrt{G}/c^2$ (see Cho 1975). We work with such a system of units that $G = c = 1$ and $\lambda = 2$.

$$\gamma_{AB} = \begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & l_{ab} \end{pmatrix} \tag{4.6}$$

where

$$l_{ab} = h_{ab} + \mu K_{ab}.$$

Let us suppose that $\det(l_{ab}) \neq 0$. Now we define on $\underline{P}$, a connection $\omega^A{}_B$ invariant with respect to the group action on $\underline{P}$ such that

$$\mathrm{D}\gamma_{A+B-} = \mathrm{D}\gamma_{AB} - \gamma_{AD} Q^D{}_{BC}(\Gamma)\theta^C = 0 \tag{4.7}$$

where $\omega^A{}_B = \Gamma^A{}_{BC}\theta^C$ and D is the exterior covariant derivative with respect to the connection $\omega^A{}_B$ (see equation (2.16)) and $Q^D{}_{BC}(\Gamma)$ is the tensor of torsion for the connection $\omega^A{}_B$. After some calculations one gets

$$\begin{aligned}
&\Gamma^\alpha_{\beta\gamma} = \bar{\Gamma}^\alpha_{\beta\gamma}, \qquad \Gamma^\mu_{\gamma b} = -l_{db} g^{\alpha\mu} L^d_{\alpha\gamma} \\
&\Gamma^\mu_{a\gamma} = -l_{ad} g^{\mu\beta}(L^d_{\gamma\beta} + 2H^d_{\beta\gamma}) \\
&\Gamma^d_{\beta\gamma} = L^d_{\beta\gamma}, \qquad \Gamma^\alpha_{\alpha c} = -g_{\alpha\rho} l^{ab} N^\rho_{cb} \\
&\Gamma^l_{c\beta} = -g_{\rho\beta} l^{ab} N^\rho_{ac}, \qquad \Gamma^\rho_{cb} = N^\rho_{cb}, \qquad \Gamma^a_{bc} = \tilde{\Gamma}^a_{bc}
\end{aligned} \tag{4.8}$$

where $L^d_{\beta\gamma}$, N^ρ_{cb}, are *ad*-type tensors on P such that

$$l_{dc} g_{\mu\beta} g^{\gamma\mu} L^d_{\gamma\alpha} + l_{cd} g_{\alpha\mu} g^{\mu\gamma} L^d_{\beta\gamma} = 2 l_{cd} g_{\alpha\mu} g^{\mu\gamma} H^d_{\beta\gamma} \tag{4.9}$$

and

$$g_{\rho\gamma} l_{db} l^{cd} N^\rho_{ca} + g_{\gamma\rho} l_{ad} l^{dc} N^\rho_{bc} = 0 \tag{4.10}$$

and $\tilde{\Gamma}^a_{bc}$ satisfies compatibility conditions

$$l_{db}\tilde{\Gamma}^d_{ac} + l_{ad}\tilde{\Gamma}^d_{cb} = \tfrac{1}{2} l_{ad} C^d_{bc}. \tag{4.11}$$

5. Geodetic equations

Let us write an equation for geodesics with respect to the connection $\omega^A{}_B$ on $\underline{P}$.

$$U^B \nabla_B U^A = 0 \tag{5.1}$$

where $U^A(t)$ is a tangent vector to the geodesic line and ∇ means covariant derivative with respect to the connection $\omega^A{}_B$. Using (4.8) one easily finds:

$$\bar{\mathrm{D}}U^\beta/\mathrm{d}t - U^b(2l_{bd} g^{\alpha\beta} H^d_{\beta\gamma} + (l_{bd} g^{\alpha\beta} L^d_{\gamma\beta} + l_{db} g^{\beta\alpha} L^d_{\beta\gamma}))U^\gamma + U^b U^c N^\alpha_{cb} = 0 \tag{5.2}$$

$$\mathrm{d}U^a/\mathrm{d}t + U^\beta U^\rho L^a_{\rho\beta} - U^\beta U^c(g_{\rho\beta} l^{ba} N^\rho_{bc} + g_{\beta\rho} l^{ab} N^\rho_{cb}) + U^b U^c \tilde{\Gamma}^a_{cb} = 0 \tag{5.3}$$

where $\bar{\mathrm{D}}/\mathrm{d}t$ means covariant derivative with respect to $\bar{\omega}^\alpha{}_\beta$ along the line to which $U^\alpha(t)$ is tangent. In the symmetric Kaluza–Klein theory (see Kerner 1968, Kopczyński 1980) $2U^b$ has the interpretation of (q^b/m_0) for a test particle (q^b is colour charge or isotopic charge of the test particle) and the system of equations (5.1) and (5.2) has first integral $U^b = \text{constant}$. In our case it is possible iff

$$L^a_{\rho\beta} = -L^a_{\beta\rho} \tag{5.4}$$

$$g_{\rho\beta} l^{ba} \cdot N^\rho_{bc} + g_{\beta\rho} l^{ab} N^\rho_{cb} = 0 \tag{5.5}$$

$$\tilde{\Gamma}^a_{cb} = -\tilde{\Gamma}^a_{bc}. \tag{5.6}$$

Using (5.5) and (4.10) one gets

$$N^\rho_{cb} = 0. \tag{5.7}$$

Finally we get

$$
\begin{aligned}
&(\bar{\mathrm{D}}U^\alpha/\mathrm{d}t) - 2U^b(l_{bd}g^{\alpha\beta}H^d_{\beta\gamma} - \tfrac{1}{2}(l_{bd}g^{\alpha\beta} - l_{db}g^{\beta\alpha})L^d_{\beta\gamma})U^\gamma = 0 \\
&\mathrm{d}U^b/\mathrm{d}t = 0
\end{aligned}
\tag{5.8}
$$

or

$$
\begin{aligned}
&\bar{\mathrm{D}}U^\alpha/\mathrm{d}t - 2U^b(l_{bd}g^{\alpha\beta}H^d_{\beta\gamma} - h_{bd}g^{[\alpha\beta]}L^d_{\beta\gamma} - \mu K_{bd}g^{(\alpha\beta)}L^d_{\beta\gamma})U^\gamma = 0 \\
&\mathrm{d}U^b/\mathrm{d}t = 0.
\end{aligned}
\tag{5.9}
$$

If the metrics $g_{\alpha\beta}$ and l_{bd} become symmetric,

$$
g^{[\alpha\beta]} = 0, \qquad K_{bd} = 0 \qquad \text{and} \qquad l_{bd} = h_{bd}
$$

one easily gets

$$
\begin{aligned}
&\bar{\mathrm{D}}U^\alpha/\mathrm{d}t - 2U^b h_{bd}g^{\alpha\beta}H^d_{\beta\gamma}U^\gamma = 0 \\
&\mathrm{d}U^b/\mathrm{d}t = 0.
\end{aligned}
\tag{5.10}
$$

(5.10) is called the Wong (see Kerner 1968 and Kopczyński 1980) equation in the case of $G = \mathrm{SU}(2)$ and contains the Lorentz force term for the Yang–Mills fields. From the historical point of view this equation should be called the Kerner equation because it appeared for the first time in Kerner's paper (1968) in curved space–time for an arbitrary semisimple gauge group. Thus in the first equation of (5.9) we obtained a Lorentz-like force term in the case of the nonsymmetric metric for an arbitrary gauge field. This term really differs from the analogous term in the symmetric Kaluza–Klein theory, but if the metric is symmetric we get the classical Kerner–Wong equation.

6. Geometry of the manifold $\underline{P}$

Using (4.8) and (5.4), (5.6) and (5.7) one easily writes the connection $\omega^A{}_B$

$$
\omega^A{}_B = \begin{pmatrix} \pi^*(\bar{\omega}^\alpha{}_\beta) - l_{db}g^{\mu\alpha}L^d_{\mu\beta}\theta^b & L^a_{\beta\gamma}\theta^\gamma \\ l_{bd}g^{\alpha\beta}(2H^d_{\gamma\beta} - L^d_{\gamma\beta})\theta^\gamma & \tilde{\omega}^a{}_b \end{pmatrix}
\tag{6.1}
$$

where

$$
L^d_{\gamma\beta} = -L^d_{\beta\gamma}
$$

is an *ad*-type tensor on P such that:

$$
\begin{aligned}
&l_{dc}g_{\mu\beta}g^{\gamma\mu}L^d_{\gamma\alpha} + l_{cd}g_{\alpha\mu}g^{\mu\gamma}L^d_{\beta\gamma} = 2l_{cd}g_{\alpha\mu}g^{\mu\gamma}H^d_{\beta\gamma} \\
&\tilde{\omega}^a{}_b = \tilde{\Gamma}^a{}_{bc}\theta^c
\end{aligned}
\tag{6.2}
$$

and

$$
l_{db}\tilde{\Gamma}^d_{ac} + l_{ad}\tilde{\Gamma}^d_{cb} = \tfrac{1}{2}l_{ad}C^d_{bc}, \qquad \tilde{\Gamma}^d_{ac} = -\tilde{\Gamma}^d_{ca}, \qquad \bar{\omega}^\alpha{}_\beta = \bar{\Gamma}^\alpha{}_{\beta\gamma}\theta^\gamma.
\tag{6.3}
$$

We define on $\underline{P}$ a second connection

$$
W^A{}_B = \begin{pmatrix} \pi^*(\bar{W}^\alpha{}_\beta) - l_{db}g^{\mu\alpha}L^d_{\mu\beta}\theta^b & L^a_{\beta\gamma}\theta^\gamma \\ l_{bd}g^{\alpha\beta}(2H^d_{\gamma\beta} - L^d_{\gamma\beta})\theta\gamma & \tilde{\omega}^a{}_b \end{pmatrix}.
\tag{6.4}
$$

Thus we have on $\underline{P}$ all $(n+4)$-dimensional analogues of quantities from Moffat's theory of gravitation i.e. $W^A{}_B$, $\omega^A{}_B$ and γ_{AB}. In this nonabelian, general case we have also the connection on a typical fibre $\tilde{\omega}^a{}_b$. This connection satisfies compatibility conditions for the nonsymmetric metric l_{ab} and is invariant with respect to the action of the group G. The last means that in the natural frame on $\underline{P}$ $\tilde{\Gamma}^a_{bc}$ is constant. The connection $\tilde{\omega}^a{}_b$ is analogous to the connection $\bar{\omega}^\alpha{}_\beta$ on space–time E. Thus we suppose that

$$\tilde{Q}^a_{ba}(\tilde{\Gamma}) = 0 \tag{6.5}$$

where $\tilde{Q}^a{}_{bc}(\tilde{\Gamma})$ is a tensor of torsion for the connection $\tilde{\omega}^a{}_b$ (cf the second equation of (4.4)). One easily finds that (6.5) is equivalent to

$$\tilde{\Gamma}^a_{ba} = 0. \tag{6.6}$$

Let us turn to calculation of a torsion for $\omega^A{}_B$.

$$\Theta^A(\Gamma) = D\theta^A. \tag{6.7}$$

One easily gets

$$Q^\alpha{}_{\beta\gamma}(\Gamma) = \bar{Q}^\alpha{}_{\beta\gamma}(\bar{\Gamma}) \tag{6.8}$$

$$\begin{aligned} Q^\alpha_{\beta b}(\Gamma) &= -Q^\alpha_{b\beta}(\Gamma) \\ &= (l_{db}g^{\gamma\alpha} + l_{bd}g^{\alpha\gamma})L^d_{\gamma\beta} - 2l_{bd}g^{\alpha\gamma}H^d_{\gamma\beta} \end{aligned} \tag{6.9}$$

$$Q^a_{\mu\nu}(\Gamma) = 2(H^a_{\mu\nu} - L^a_{\mu\nu}) \tag{6.10}$$

$$Q^a_{bc}(\Gamma) = \tilde{Q}^a_{bc}(\tilde{\Gamma}) = -(\tfrac{1}{2}C^a_{bc} + 2\tilde{\Gamma}^a_{bc}) \tag{6.11}$$

where $\bar{Q}^\alpha_{\beta\gamma}(\bar{\Gamma})$ is the tensor of torsion for the connection $\bar{\omega}^\alpha{}_\beta$ and $\tilde{Q}^a_{bc}(\tilde{\Gamma})$ is the tensor of torsion for the connection $\tilde{\omega}^a_b$.

Let us define the *ad*-type tensor on $\underline{P}$, $K^a_{\mu\nu}$ such that

$$L^a_{\mu\nu} = H^a_{\mu\nu} + K^a_{\mu\nu}. \tag{6.12}$$

We have

$$Q^a_{\mu\nu}(\Gamma) = -2K^a_{\mu\nu}. \tag{6.13}$$

We will find the physical interpretation of this tensor. Let us turn to the calculation of the 2-form of curvature for $\omega^A{}_B$.

$$\Omega^A{}_B(\Gamma) = \mathrm{d}\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B. \tag{6.14}$$

After some calculations one finds using (6.1)

$$\begin{aligned} \Omega^\alpha_\beta(\Gamma) = \bar{\Omega}^\alpha_\beta(\bar{\Gamma}) &+ [l_{de}g^{\alpha\rho}(2H^e_{\rho[\gamma} - L^e_{\rho[\gamma})L^d_{\nu]\beta} - l_{bd}g^{\mu\alpha}L^d_{\mu\beta}H^b_{\gamma\nu}] \\ &\times \theta^\gamma \wedge \theta^\nu - \overset{\text{gauge}}{\bar{\nabla}}_\rho[l_{bd}g^{\mu\alpha}L^d_{\mu\beta}]\theta^\rho \wedge \theta^b + l_{d[b}l_{|a|c}g^{\mu\alpha}g^{\nu\delta}L^d_{\mu\delta}L^a_{\nu\beta}\theta^b \wedge \theta^c \end{aligned} \tag{6.15a}$$

$$\begin{aligned} \Omega^\alpha_b(\Gamma) = &\Big(\overset{\text{gauge}}{\bar{\nabla}}_{[\rho}[l_{bd}g^{\alpha\beta}(2H^d_{\gamma]\beta} - L^d_{\gamma]\beta})] + \tfrac{1}{2}l_{bd}g^{\alpha\beta}(2H^d_{\mu\beta} - L^d_{\mu\beta})\bar{Q}^\mu{}_{\rho\gamma}(\bar{\Gamma})\Big)\theta^\rho \wedge \theta^\gamma \\ &- l_{fa}l_{bd}g^{\mu\alpha}g^{\delta\beta}L^f_{\mu\delta}(2H^d_{\gamma\beta} - L^d_{\gamma\beta})\theta^a \wedge \theta^\gamma \end{aligned} \tag{6.15b}$$

$$\Omega^a{}_\beta(\Gamma) = \Big(\overset{\text{gauge}}{\bar{\nabla}}_{[\rho}L^a_{|\beta|\gamma]} + \tfrac{1}{2}L^a_{\beta\mu}\bar{Q}^\mu_{\rho\gamma}(\bar{\Gamma})\Big)\theta^\rho \wedge \theta^\gamma + (\tilde{\nabla}_b L^a_{\beta\gamma} + l_{cb}g^{\mu\delta}L^c_{\mu\beta}L^a_{\delta\gamma})\theta^b \wedge \theta^\gamma \tag{6.15c}$$

$$\Omega^a_b(\Gamma) = \tilde{\Omega}^a_b(\tilde{\Gamma}) + l_{bf} g^{\delta\beta} L^a_{\delta[\gamma}(2H^f_{\rho]\beta} - L^f_{\rho]\beta})\theta^\gamma \wedge \theta^\rho \tag{6.15d}$$

where $\bar{\Omega}^\alpha_\beta(\bar{\Gamma})$ is the 2-form of curvature $\underset{\text{gauge}}{\text{for}}$ the connection $\bar{\omega}^\alpha{}_\beta$, $\tilde{\Omega}^a_b(\tilde{\Gamma})$ is the 2-form of curvature for the connection $\tilde{\omega}^a_b$. $\bar{\nabla}_\mu$ means 'gauge' derivative *and* generally covariant derivative with respect to the connection $\bar{\omega}^\alpha{}_\beta$ at once. $\tilde{\nabla}_b$ means covariant derivative with respect to the connection $\tilde{\omega}^a_b$ on a typical fibre. $\bar{Q}^\mu_{\gamma\rho}(\bar{\Gamma})$ is the tensor of torsion for the connection $\bar{\omega}^\alpha{}_\beta$. One easily finds using the third equation of (2.13) the tensor of curvature for $\omega^A{}_B$ from (6.15*a*–*d*).

$$R^\alpha{}_{\beta\gamma\nu}(\Gamma) = \bar{R}^\alpha_{\beta\gamma\nu}(\bar{\Gamma}) + 2[l_{de} g^{\alpha\rho}(2H^e_{\rho[\gamma} - L^e_{\rho[\gamma})L^d_{\nu]\beta} - l_{bd} g^{\mu\alpha} L^d_{\mu\beta} H^b_{\gamma\nu}] \tag{6.16a}$$

$$R^\alpha{}_{\beta\rho b}(\Gamma) = -R^\alpha{}_{\beta b\rho}(\Gamma) = -\overset{\text{gauge}}{\bar{\nabla}}_\rho (l_{bd} g^{\mu\alpha} L^d_{\mu\beta}) \tag{6.15b}$$

$$R^\alpha{}_{\beta bc}(\Gamma) = 2 l_{f[b} l_{|a|c} g^{\mu\alpha} g^{\nu\delta} L^f_{\mu\delta} L^a_{\nu\beta} \tag{6.16c}$$

$$R^\alpha{}_{b\rho\gamma}(\Gamma) = 2\Big(\overset{\text{gauge}}{\bar{\nabla}}_{[\rho}[l_{bd} g^{\alpha\beta}(2H^d_{\gamma]\beta} - L^d_{\gamma]\beta})]\Big) + \tfrac{1}{2} l_{bd} g^{\alpha\beta}(2H^d_{\mu\beta} - L^d_{\mu\beta})\bar{Q}^\mu_{\rho\gamma}(\bar{\Gamma}) \tag{6.16d}$$

$$R^\alpha{}_{ba\gamma}(\Gamma) = -R^\alpha{}_{b\gamma a}(\Gamma) = -l_{fa} l_{bd} g^{\mu\alpha} g^{\delta\beta} L^f_{\mu\delta}(2H^d_{\gamma\beta} - L^d_{\gamma\beta}) \tag{6.16e}$$

$$R^a{}_{\beta\rho\gamma}(\Gamma) = 2\Big(\overset{\text{gauge}}{\bar{\nabla}}_{[\rho} L^a_{|\beta|\gamma]} + \tfrac{1}{2} L^a_{\beta\mu} \bar{Q}^\mu{}_{\rho\gamma}(\bar{\Gamma})\Big) \tag{6.16f}$$

$$R^a{}_{\beta b\gamma}(\Gamma) = -R^a{}_{\beta\gamma b}(\bar{\Gamma}) = \tilde{\nabla}_b L^a_{\beta\gamma} + l_{cb} g^{\mu\delta} L^c_{\mu\beta} L^a_{\delta\gamma} \tag{6.16g}$$

$$R^a{}_{bcd}(\Gamma) = \tilde{R}^a{}_{bcd}(\tilde{\Gamma}) \tag{6.16h}$$

$$R^a{}_{b\gamma\rho}(\Gamma) = 2 l_{bf} g^{\delta\beta} L^a_{\delta[\gamma}(2H^f_{\rho]\beta} - L^f_{\rho]\beta}) \tag{6.16i}$$

$$R^a{}_{\beta cd}(\Gamma) = R^a{}_{bc\rho}(\Gamma) = R^a_{b\rho c}(\Gamma) = 0. \tag{6.16j}$$

Let us turn to calculations of the Moffat–Ricci tensor for $\omega^A{}_B$ and the Moffat–Ricci curvature scalar. We have

$$R_{BC}(\Gamma) = R^A{}_{BCA}(\Gamma) + \tfrac{1}{2} R^A{}_{ABC}(\Gamma) \tag{6.17}$$

(see for example Moffat 1982). Thus we have

$$R_{\beta\gamma}(\Gamma) = R^\alpha{}_{\beta\gamma\alpha}(\Gamma) + \tfrac{1}{2} R^\alpha{}_{\alpha\beta\gamma}(\Gamma) + R^a{}_{\beta\gamma a} + \tfrac{1}{2} R^a{}_{a\beta\gamma} \tag{6.18}$$

$$R_{bc}(\Gamma) = R^\alpha{}_{bc\alpha}(\Gamma) + \tfrac{1}{2} R^\alpha{}_{\alpha bc}(\Gamma) + R^a{}_{bca} + \tfrac{1}{2} R^a{}_{abc} \tag{6.19}$$

and

$$R(\Gamma) = \gamma^{BC} R_{BC}(\Gamma) = g^{B\gamma} R_{\beta\gamma}(\Gamma) + l^{bc} R_{bc}(\Gamma). \tag{6.20}$$

After some calculations one finds

$$R(\Gamma) = \bar{R}(\bar{\Gamma}) + \tilde{R}(\tilde{\Gamma}) - l_{ab}(2H^a \cdot H^b - L^{a\mu\nu} H^b_{\mu\nu}) \tag{6.21}$$

where $\bar{R}(\bar{\Gamma})$ is the Moffat–Ricci curvature scalar for the connection $\bar{\omega}^\alpha{}_\beta$, $\tilde{R}(\tilde{\Gamma})$ is the Moffat–Ricci curvature scalar for the connection $\tilde{\omega}^a{}_b$,

$$l_{ab} = h_{ab} + \mu K_{ab} \tag{6.22}$$

$$L^{a\mu\nu} = g^{\alpha\mu} g^{\beta\nu} L^a_{\alpha\beta} \tag{6.23}$$

$$H^a = g^{[\alpha\beta]} H^a{}_{\alpha\beta}. \tag{6.24}$$

Now we pass to calculation of the Moffat–Ricci curvature scalar for $W^A{}_B$. It is very easy to see that

$$R(W)=\bar{R}(\bar{W})+\tilde{R}(\tilde{\Gamma})-l_{ab}(2H^a\cdot H^b-L^{a\mu\nu}H^b_{\mu\nu}) \tag{6.25}$$

where $\bar{R}(\bar{W})$ is the Moffat–Ricci curvature scalar for the connection $\bar{W}^\alpha{}_\beta$.

7. The connection $\tilde{\omega}^a{}_b$: cosmological constant

Let us turn to the calculation of the Moffat–Ricci curvature scalar for the connection $\tilde{\omega}^a{}_b:\tilde{R}(\tilde{\Gamma})$. One can find using (6.3) and (6.6)

$$\tilde{R}(\tilde{\Gamma})=l^{bd}\tilde{\Gamma}^a_{cd}\tilde{\Gamma}^c_{ba}-\tfrac{1}{2}l^{bd}\tilde{\Gamma}^a_{bc}C^c_{da} \tag{7.1}$$

and using (6.3) and (6.6) one finally gets

$$\tilde{R}(\tilde{\Gamma})=-h_{pb}l^{ac}l^{rf}\tilde{\Gamma}^p_{cr}\tilde{\Gamma}^b_{fa} \tag{7.2}$$

where $\tilde{\Gamma}^a_{bc}$ satisfies compatibility conditions

$$l_{db}\tilde{\Gamma}^d_{ac}(\mu)+l_{ad}\tilde{\Gamma}^d_{cb}(\mu)=\tfrac{1}{2}l_{ad}C^d_{bc} \tag{7.3}$$

and

$$\tilde{\Gamma}^b_{ac}(\mu)=-\tilde{\Gamma}^b_{ca}(\mu), \qquad \tilde{\Gamma}^b_{ab}(\mu)=0. \tag{7.4}$$

It is easy to see that $\tilde{R}(\tilde{\Gamma})$ is a rational function of μ. But it is a very difficult task to find the exact dependence on μ. Therefore, we have not an exact solution of (7.3) and (7.4), but we can find an asymptotic dependence for very large μ. If $\mu\to\infty$ (7.3) becomes

$$K_{db}\tilde{\Gamma}^d_{ac}+K_{ad}\tilde{\Gamma}^d_{cb}=\tfrac{1}{2}K_{ad}C^d_{bc}. \tag{7.5}$$

Thus in the limit of very large μ, $\tilde{\Gamma}^a_{bc}(\mu)$ goes to the constant $\tilde{\Gamma}^a_{bc}$ with respect to μ. On the other hand we have

$$l^{ab}=\Delta^{ab}/\Delta \tag{7.6}$$

where $\Delta=\det(l_{ab})$ and Δ^{ab} is a cofactor matrix formed from l_{ab}. It is easy to see that Δ is a polynomial of nth order with respect to μ and Δ^{ab} a polynomial of $(n-1)$th order with respect to μ. Thus finally we get for very large μ

$$\tilde{R}(\tilde{\Gamma})=-h_{pb}\Delta^{ac}\Delta^{rf}\tilde{\Gamma}^p_{cr}\tilde{\Gamma}^b_{fa}/\Delta^2\sim\text{constant}/\mu^2. \tag{7.7}$$

If μ is sufficiently large, $\tilde{R}(\tilde{\Gamma})$ is as small as we want. It is very important, because in the classical (symmetric) Kaluza–Klein theory $\tilde{R}(\tilde{\Gamma})$ plays the role of the cosmological constant. This cosmological constant is 10^{127} bigger than the upper limit from observational data. Now we are able to make the cosmological constant as small as we need. In this way we get the physical interpretation of dimensionless parameter μ and some limits imposed on it

$$|\mu|\geqslant 10^{64}. \tag{7.8}$$

Maybe it is possible to find an exact solution of (7.3) and (7.4). In this way we get

$$\tilde{R}(\tilde{\Gamma})=P_m(\mu)/Q_{m+2}(\mu) \tag{7.9}$$

where P_m and Q_{m+2} are some polynomials with respect to μ of order m and $(m+2)$.

If the polynomial $P_m(\mu)$ has a real root μ_0 we have

$$\tilde{R}(\tilde{\Gamma})=0 \qquad \text{for } \mu=\mu_0.$$

If we suppose that $\tilde{\Gamma}^f_{cb}$ has a potential Ξ^f_d such that

$$\tilde{\Gamma}^f=\tfrac{1}{2}\tilde{\Gamma}^f_{cb}\theta^c\wedge\theta^b=\tilde{d}\Xi^f=\tilde{d}(\Xi^f_d\theta^d) \tag{7.10}$$

($\tilde{d}$ means here the vertical part of d, $\tilde{d}\Xi^f=d\Xi^f-\text{hor}\, d\Xi^f$, and Ξ^f_d, is constant in the frame θ^c). We can transform (7.3) into

$$\Xi^d_e(l_{db}C^e_{ac}+l_{ad}C^e_{cb})=-l_{ad}C^d_{bc} \tag{7.11}$$

and we have

$$\tilde{\Gamma}^f_{mn}=-\tfrac{1}{2}\Xi^f_dC^d_{mn}. \tag{7.12}$$

It is easy to see that

$$\tilde{\Gamma}^f_{mn}=-\tilde{\Gamma}^f_{nm} \tag{7.13}$$

and from the second equation of (7.4) we get:

$$\Xi^f_dC^d_{fm}=0. \tag{7.14}$$

In the terms of the potential Ξ^f_d the Moffat–Ricci curvature scalar turns into:

$$\tilde{R}(\tilde{\Gamma})=-\tfrac{1}{4}h_{pb}(l^{ac}C^n_{crl}l^{rf}C^m_{fa})(\Xi^p_n\Xi^d_m). \tag{7.15}$$

If the metric l_{ab} is symmetric, one easily finds from (7.11)

$$\Xi^d_e=-\tfrac{1}{2}\delta^d_e. \tag{7.16}$$

Unfortunately the general solution in the nonsymmetric case is unknown.

8. The variational principle and field equations. Interpretations and conclusions

Let us define the Palatini variational principle on the manifold $\underset{\sim}{P}$ for $R(W)$

$$\delta\int_V R(W)\sqrt{\gamma}\,\mathrm{d}^{n+4}X=0, \qquad V\subset \underset{\sim}{P} \tag{8.1}$$

where $\gamma=\det(\gamma_{AB})=\det(g_{\alpha\beta})\cdot\det(l_{ab})=g\cdot\Delta$. We vary with respect to independent quantities: $g_{\alpha\beta}$, $\bar{W}^\alpha_{\beta\gamma}$ and ω^a. After simple calculations one gets

$$\bar{R}_{\alpha\beta}(\bar{W})-\tfrac{1}{2}g_{\alpha\beta}\bar{R}(\bar{W})=8\pi(\overset{\text{gauge}}{T}_{\alpha\beta}+\rho\cdot g_{\alpha\beta}) \tag{8.2}$$

$$\mathbf{g}^{[\mu\nu]}{}_{,\nu}=0 \tag{8.3}$$

$$g_{\mu\nu,\sigma}-g_{\zeta\nu}\bar{\Gamma}^\zeta_{\mu\sigma}-g_{\mu\zeta}\bar{\Gamma}^\zeta_{\sigma\nu}=0 \tag{8.4}$$

$$\overset{\text{gauge}}{\nabla}_\mu(\boldsymbol{L}^{a\alpha\mu})=4\mathbf{g}^{[\alpha\beta]}\overset{\text{gauge}}{\nabla}_\beta(g^{[\mu\nu]}H^a_{\mu\nu}) \tag{8.5}$$

where

$$\overset{\text{gauge}}{T}_{\alpha\beta} = (l_{ab}/4\pi)[(g^{\gamma\mu}L^{a}_{\gamma\alpha}H^{b}_{\mu\beta} - 2g^{[\mu\nu]}H^{a}_{\mu\nu}H^{b}_{\alpha\beta} - \tfrac{1}{4}g_{\alpha\beta})L^{a\mu\nu}H_{\mu\nu} - 2(g^{[\mu\nu]}H^{a}_{\mu\nu})(g^{[\gamma\delta]}H^{b}_{\gamma\delta})] \tag{8.6}$$

$$\mathfrak{g}^{[\mu\nu]} = \sqrt{-g}\, g^{[\mu\nu]}, \qquad \mathfrak{L}^{a\mu\nu} = \sqrt{-g}\, g^{\beta\mu} g^{\gamma\alpha} L^{a}_{\beta\gamma} \tag{8.7}$$

$\overset{\text{gauge}}{\nabla}_{\mu}$ means gauge derivative

$$l_{dc}g_{\mu\beta}g^{\gamma\mu}L^{d}_{\gamma\alpha} + l_{cd}g_{\alpha\mu}g^{\mu\gamma}L^{d}_{\beta\gamma} = 2l_{cd}g_{\alpha\mu}g^{\mu\gamma}H^{d}_{\beta\gamma} \tag{8.8}$$

and

$$\rho = \rho(\mu) = \tfrac{1}{16}\pi\tilde{R}(\tilde{\Gamma}). \tag{8.9}$$

The equations (8.2) and (8.3) are equations for the gravitational field in the presence of Yang–Mills field (gauge) sources. $\overset{\text{gauge}}{T}_{\alpha\beta}$ plays the role of an energy–momentum tensor for the gauge (Yang–Mills) field. Equation (8.4) is a compatibility condition for the metric on space–time (see (4.4)). Equation (8.5) plays the role of the second Yang–Mills equation. It is easy to see that

$$g^{\alpha\beta}\overset{\text{gauge}}{T}_{\alpha\beta} = 0. \tag{8.10}$$

Now we are able to interpret all quantities in our theory. First of all it is easy to see that $L^{a}_{\alpha\beta}$ plays the role of the second tensor of the Yang–Mills field (gauge) strength and (equation (8.8)) expresses the relationship between both tensors $H^{a}_{\alpha\beta}$ and $L^{a}_{\alpha\beta}$.

In the electromagnetic case ($G = \mathrm{U}(1)$) we have the tensors $F_{\alpha\beta}$ and $H_{\alpha\beta}$ (see Kalinowski 1982) which are the first and second tensors of electromagnetic strength.

In the classical electrodynamics of continuous media (de Groot *et al* 1972) or in nonlinear electrodynamics (Plebański 1970) it is necessary to define both of these tensors. The first tensor $F_{\alpha\beta}$ is built from $(\boldsymbol{E}, \boldsymbol{B})$, the second from $(\boldsymbol{D}, \boldsymbol{H})$.

Here we build $H^{a}_{\alpha\beta}$ from $(\boldsymbol{E}^{a}, \boldsymbol{B}^{a})$ and $L^{a}_{\alpha\beta}$ from $(\boldsymbol{D}^{a}, \boldsymbol{H}^{a})$. For example in quantum chromodynamics we have $\boldsymbol{D}^{a}$ (Nielsen and Patkos 1982). The vacuum behaves as a dielectric for gluon fields. If the metrics $g_{\alpha\beta}$ and l_{ab} are symmetric, $H^{a}_{\alpha\beta} = L^{a}_{\alpha\beta}$. Thus it is interesting that the skewsymmetric part of the metric γ_{AB} induces some kind of Yang–Mills field polarisation tensor of the vacuum. In the electromagnetic case (Kalinowski 1982) ($G = \mathrm{U}(1)$) we define the electromagnetic polarisation tensor of the vacuum $M_{\alpha\beta}$ induced by the skewsymmetric part of the metric such that

$$H_{\alpha\beta} = F_{\alpha\beta} - 4\pi M_{\alpha\beta} \tag{8.11}$$

($L^{a}_{\alpha\beta}$ is analogous to $H_{\alpha\beta}$ and $H^{a}_{\alpha\beta}$ to $F_{\alpha\beta}$).

In the classical electrodynamics of continuous media (de Groot *et al* 1972) or in nonlinear electrodynamics (Plebański 1970) this tensor is usually defined. Here we can define the tensor $M^{a}_{\alpha\beta}$ such that

$$L^{a}_{\alpha\beta} = H^{a}_{\alpha\beta} - 4\pi M^{a}_{\alpha\beta} \tag{8.12}$$

where $M^{a}_{\alpha\beta}$ is the Yang–Mills field analogue of the electromagnetic polarisation tensor $M_{\alpha\beta}$. It is easy to see that

$$4\pi M^{a}_{\alpha\beta} = -K^{a}_{\alpha\beta} \tag{8.13}$$

(see (6.13)). Thus we get a geometrical interpretation of $M^a_{\alpha\beta}$

$$Q^a_{\alpha\beta}(\Gamma) = 8\pi M^a_{\alpha\beta} \tag{8.14}$$

($M^a_{\alpha\beta}$ is of course the *ad*-type tensor defined on $\underline{P}$). Thus the Yang–Mills field polarisation induced by the skewsymmetric part of the metric γ_{AB} is the torsion in the additional dimensions. This is in very good accordance with results from Kalinowski (1981). The only difference is that there the Yang–Mills field polarisation has its origin from external sources and (8.13) plays the role of the Cartan equation in the Kaluza–Klein theory with torsion. But this is not all. The skewsymmetric part of the metric γ_{AB} also changes the Yang–Mills field Lagrangian

$$\mathscr{L}_{\mathrm{YM}} = -(l_{ab}/8\pi)[2(g^{[\alpha\beta]}H^a_{\alpha\beta})(g^{[\mu\nu]}H^b_{\mu\nu}) - L^{a\mu\nu}H^b_{\mu\nu}]. \tag{8.15}$$

In (8.15) we have a new term

$$-2h_{ab}(g^{[\alpha\beta]}H^a_{\alpha\beta})(g^{[\mu\nu]}H^b_{\mu\nu})$$

which is an interaction between the skewon field and the Yang–Mills field. This term vanishes if the metric of space–time is symmetric and is always non-negative if the group G is compact. The second term in (8.15) is also a little different from in the classical Yang–Mills field Lagrangian. In the place of the symmetric tensor h_{ab} we now have the nonsymmetric tensor

$$l_{ab} = h_{ab} + \mu K_{ab}.$$

The skewsymmetric part of the metric induces also a source for the Yang–Mills field. In equation (8.5) we get a current

$$J^{\alpha a} = \frac{1}{\pi} g^{[\alpha\beta]} \overset{\text{gauge}}{\nabla}_\beta (g^{[\mu\nu]}H^a_{\mu\nu}). \tag{8.16}$$

This current vanishes if the metric is symmetric. This is completely different from in the classical Kaluza–Klein theory (see Kerner 1968, Cho 1975 and Kalinowski 1983). In the classical (symmetric) approach based on a symmetric metric on $\underline{P}$ one obtained the second Yang–Mills equation in the vacuum. Thus the nonsymmetric Kaluza–Klein theory, combining Moffat's theory and the Yang–Mills (gauge field) theory, is stronger than the classical Kaluza–Klein approach combining general relativity and a gauge theory. In the nonsymmetric Kaluza–Klein theory there exist 'interference effects' between gravitation and gauge fields which are absent in the classical approach (neglecting the appearance of the cosmological constant which is a disadvantage of the theory and is possible to remove in some approaches (Kalinowski 1983)). These new 'interference effects' are the following.

(i) The new term in the Yang–Mills Lagrangian

$$(1/4\pi)h_{ab}(g^{[\mu\nu]}H^a_{\mu\nu})(g^{[\alpha\beta]}H^b_{\alpha\beta}).$$

(ii) The change in the classical part of the Yang–Mills field Lagrangian in replacing h_{ab} by l_{ab}.

(iii) The existence of a Yang–Mills field polarisation of the vacuum $M^a_{\alpha\beta}$ which has geometrical interpretation as a torsion in the additional dimensions.

(iv) The additional term in the Kerner–Wong equation (equation of motion for the test particle in the gravitational and Yang–Mills fields)

$$\tfrac{1}{2}(q^b/m_0)(l_{bd}g^{\alpha\beta} - l_{db}g^{\beta\alpha})L^d_{\beta\gamma}U^\gamma.$$

(v) The existence of the cosmological constant $\rho(\mu)$ with asymptotic behaviour for large μ

$$\rho(\mu) \sim \text{constant}/\mu^2.$$

Due to these five fundamental 'interference effects' we get other effects.

(i) The new energy–momentum tensor $\overset{\text{gauge}}{T_{\alpha\beta}}$ with zero trace.

(ii) Sources for Yang–Mills fields the current $J^{\alpha a}$.

All of these 'interference effects' vanish if the metric of $\underset{\sim}{P}$ becomes symmetric. In this case we get classical Kaluza–Klein theory.

9. Special cases

Let us consider some special cases of the theory. First of all let $g_{\alpha\beta}$ be symmetric and $l_{ab} \neq l_{ba}$. In this case we are able to solve equation (8.8) and we get:

$$L^a_{\beta\gamma} = h^{ac} l_{cd} H^d_{\beta\gamma}. \tag{9.1}$$

The Yang–Mills Lagrangian takes the form

$$\mathscr{L}_{\text{YM}} = (1/8\pi)(h_{bd} + \mu^2 K^c{}_b K_{cd}) H^{b\mu\nu} H^d_{\mu\nu} \tag{9.2}$$

where $K^c{}_d = h^{ce} K_{ed}$. Let us suppose that $l_{ab} = h_{ab}$ and $g_{\alpha\beta} \neq g_{\beta\alpha}$; in this case we have

$$\mathscr{L}_{\text{YM}} = (1/8\pi) h_{ab}(H^a H^b - L^{a\mu\nu} H^b_{\mu\nu}) \tag{9.3}$$

where $H^a = g^{[\mu\nu]} H^a_{\mu\nu}$ and the relationship between $L^a_{\alpha\beta}$ and $H^a_{\alpha\beta}$ is the following

$$g_{\mu\beta} g^{\gamma\mu} L^a_{\gamma\alpha} + g_{\alpha\mu} g^{\mu\gamma} L^a_{\beta\gamma} = 2 g_{\alpha\mu} g^{\mu\gamma} H^a_{\beta\gamma}. \tag{9.4}$$

Now there is not mixing in the gauge indices (not mixing of 'colour charges'). In the first special case we are able to calculate the polarisation tensor $M^a_{\alpha\beta}$ and we get

$$M^a_{\alpha\beta} = (1/4\pi)(\delta^a_d - h^{ac} l_{cd}) H^d_{\alpha\beta}. \tag{9.5}$$

In the first case we are able to make the cosmological constant as small as we need. In the second case we get the classical result with enormous cosmological constant. If $G = \text{U}(1)$ we get the results from Kalinowski (1983)

$$\mathscr{L}_{em} = \frac{1}{8\pi}[2(g^{[\mu\nu]} F_{\mu\nu})^2 - H^{\mu\nu} F_{\mu\nu}] \tag{9.6}$$

where

$$g_{\mu\beta} g^{\gamma\mu} H_{\gamma\beta} + g_{\alpha\mu} g^{\mu\gamma} H_{\beta\gamma} = 2 g_{\alpha\mu} g^{\mu\gamma} F_{\beta\gamma} \tag{9.7}$$

and we do not obtain the cosmological constant (U(1) is abelian).

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